Feb 1

Ricci flow. $\frac{\partial g}{\partial t} = -2 Ric(g)$

metric destinguished by its curvature

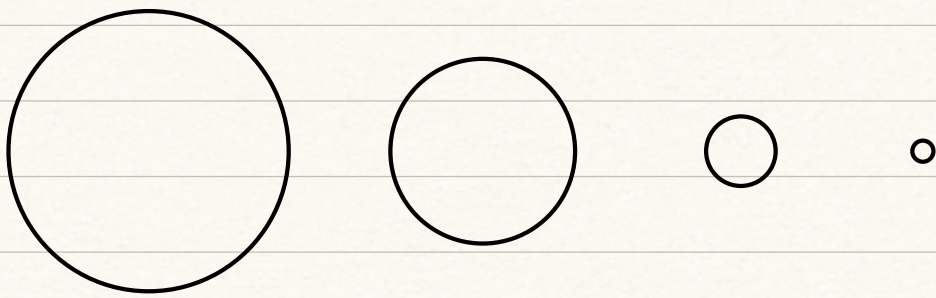
cause singularity.

surgery. excising sing region

Ric $\longrightarrow$ Laplacian in harmonic coords

conn. Laplacian "rough" Laplacian

Laplace - Beltrami $f : (M, g) \longrightarrow \mathbb{R}$



Example 1.

IC $t=0$ $Ric(g_0) = \lambda g_0$

$$\Rightarrow \quad g(t) = (1 - 2\lambda t)\, g_0$$

$\lambda > 0$ collapse at $T = \frac{1}{2\lambda}$

$\lambda < 0$ expand

(1.2.3) $\quad \psi_t^*(g(t)) := g \circ \psi_t(t)$

$$\mathcal{L}_X g(s) = \frac{\psi_t^* g(s) - \psi_0^* g(s)}{t-0}$$

$$\frac{\psi_t^*(g(t) - g(s)) + \psi_t^*(g(s)) - g(s)}{t} = \psi_t^*\left(\frac{\partial g}{\partial t}\right) + \psi_t^*(\mathcal{L}_X g)$$

Example 2. Ricci soliton

$$\hat{g}(t) = \sigma(t)\, \psi_t^*(g(t))$$

in particular

$$\hat{g}(t) = (1 - 2\lambda t)\, \psi_t^* g_0 \quad \text{with}$$

$$g_0 : -Ric(g_0(t)) = \mathcal{L}_Y g_0 - 2\lambda g_0 \qquad IC$$

$Y = \sigma(t) X.$

$\sim$ defines a flow ψ_t

<0 expanding

$\lambda = 0$ steady

>0 shrinking

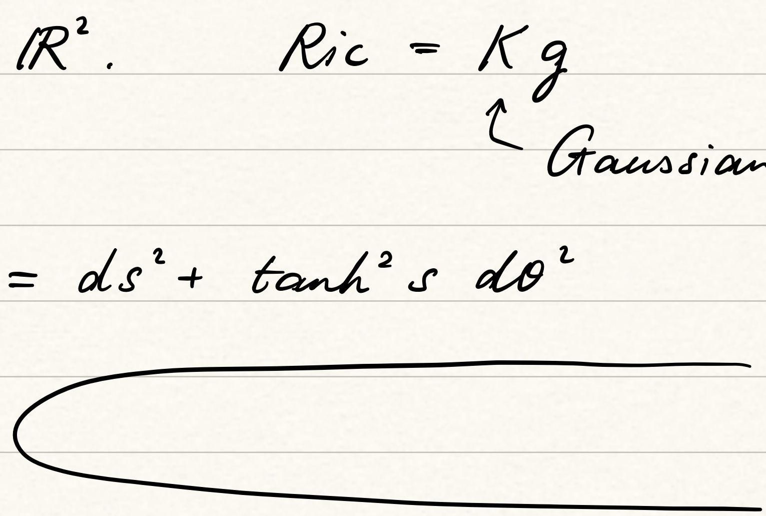
Example 3. steady Ricci soliton

Hamilton's cigar / Witten's black hole

$M = \mathbb{R}^2$. $\quad Ric = Kg \quad$ at $t=0$

Gaussian $\quad -\frac{1}{\rho^2} \Delta \ln \rho$

$$g_0 = ds^2 + \tanh^2 s \, d\theta^2 \qquad K = \frac{2}{\cosh^2 s}$$



parabolic rescalling $\quad \hat{g}(x,t) = \lambda g(x, \frac{t}{\lambda})$

pass to the limit $\quad \hat{R} = \frac{1}{\lambda} R$

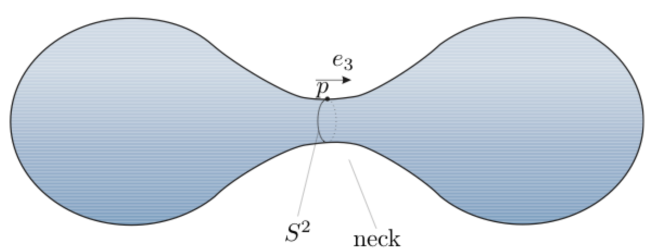


Figure 1.2: 3-sphere

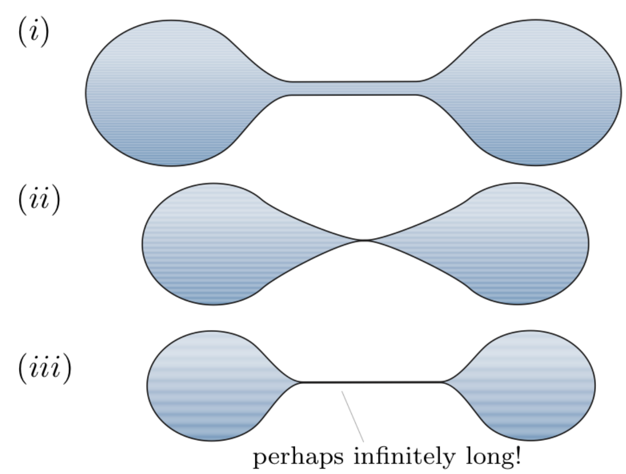


Figure 1.3: Neck Pinch.

singularity

large curvature $\xrightarrow[\text{magnify}]{\text{blow up}}$ small curvature

$\mathcal{M}$ compact $\quad$ $\mathrm{int}(\mathcal{M}) = X/\Gamma$
finite vol.

2D. arbitary metric $\longrightarrow$ metric of const curvature
within the conf. class.

Thurson's geometrisation conj.

$$\frac{\partial g}{\partial t} = h = -2\,Ric(g)$$

$$\Rightarrow \quad \frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2 \geq \Delta R + \frac{2}{n} R^2.$$

f under Ricci flow

$$\frac{\partial}{\partial t} \Delta f = \Delta \frac{\partial f}{\partial t} + 2\langle Ric, Hess(f)\rangle$$

$$\frac{d\,Vol}{dt} = -\int R \, dV$$

Ch 3. Maximum principle

$\boxed{\text{weak, scalar}}$

- $\dfrac{\partial u}{\partial t} \le \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u,t)$

- $\begin{cases} \dfrac{d\phi_\varepsilon}{dt} = F(\phi_\varepsilon(t), t) + \varepsilon \\ \phi_\varepsilon(0) = \alpha + \varepsilon \in \mathbb{R} \end{cases}$

- $u(\cdot, 0) \le \alpha$

$\Rightarrow \quad u(\cdot, t) \le \phi(t) \qquad \forall\, t \in [0, T]$

(Max principle) + (evolution of curvature) $\rightsquigarrow$ (behaviour of Rm and R)

Curvature control

Cor 2.5.5 gives $\bullet$ $\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n} R^2$

- $\begin{cases} \frac{d\phi}{dt} = \frac{d}{dt}\left(\frac{\alpha}{1-\frac{2\alpha}{n}t}\right) = \frac{\alpha \cdot \frac{2\alpha}{n}}{\left(1-\frac{2\alpha}{n}t\right)^2} \\ \phi(0) = \alpha \end{cases}$

$$= \frac{2}{n}\phi^2 = F(\phi, t).$$

- $R(0) \geq \alpha$

max principle $\Rightarrow$ $R \geq \frac{\alpha}{1-\frac{2\alpha}{n}t}$

Volume control

closed manifold M

$t \in [0, T]$

- $R(g_0) \geq 0$ $\Rightarrow$ $Vol(g_t) \searrow$ weakly
- $\alpha = \inf R(g_0) \leq 0$ $\Rightarrow$

$$\frac{V(t)}{\left(1+\frac{2(-\alpha)}{n}t\right)^{\frac{n}{2}}} \searrow \text{ weakly}$$

$$V(t) \leq V(0)\left(1+\frac{2(-\alpha)}{n}t\right)^{\frac{n}{2}}$$

- $\lim \frac{V(t)}{\left(1+\frac{2(-\alpha)}{n}t\right)^{\frac{n}{2}}} = \bar{V}(t)$ exists.

$= 0$ graph

> 0 hyperbolic

Ricci flow preserves +ive R

$|Rm|$ control

$$\frac{\partial}{\partial t}|Rm|^2 \le \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3$$

$$\le \Delta |Rm|^2 + C|Rm|^3$$

max. principle

$$|Rm(g_0)| \le M \implies |Rm| \le \frac{M}{1-\frac{1}{2}CMt}$$

Derivative estimate (global on $M \times [0, \frac{1}{M}]$)

curvature satisfies heat eqn. elliptic

$\longrightarrow$ control higher order derivatives by l.o. derivatives

$$|Rm| \le M \quad \text{on} \quad M \times [0, \tfrac{1}{M}]_t$$

$$\Rightarrow |\nabla^k Rm| \le \frac{C(n,k)M}{t^{k/2}}$$

$$\left|\frac{\partial^j}{\partial t^j}\nabla^k Rm\right| \le \frac{C(n,k)M}{t^{j+k/2}}$$

Ch 4 Parabolic PDE I. linear

On $\Omega \subseteq \mathbb{R}^n$, $u : \Omega \longrightarrow \mathbb{R}$

$$L(u) = \underbrace{a_{ij}}_{\text{unif. +ive definite}} \partial_i \partial_j u + b_i \partial_i u + c$$

On manifolds $\mathcal{M}$, $u : \mathcal{M} \longrightarrow \mathbb{R}$

$$L(u) = a^{ij} \partial_i \partial_j u + b^i \partial_i u + c$$

principal symbol

$$\sigma(L)(x, \xi) f(x) = \lim_{s \to \infty} s^{-2} e^{-s\phi(x)} L(e^{s\phi} f)(x)$$

eliminate s^2 due to $\partial_i \partial_j$

Require: $\sigma(L)\underbrace{(x, \xi)}_{\in T^*\mathcal{M}} \geq \lambda |\xi|^2$

$$\begin{cases} \partial_t u = L(u) & \text{on } \mathcal{M} \times [0, \infty) \\ u(0) = u_0 & \text{on } \mathcal{M} \end{cases}$$

$\exists!$ finite sol.

On vector bundle $E \xrightarrow{\pi} \mathcal{M}$, $v : \mathcal{M} \longrightarrow E$

$$L(v) = \left(a^{ij}_{\alpha\beta}\, \partial_i \partial_j v^\beta + b^i_{\alpha\beta}\, \partial_i v^\beta + c_{\alpha\beta} v^\beta \right) e_\alpha$$

$$\sigma(L)(x, \xi) v = \left(a^{ij}_{\alpha\beta}\, \xi_i \xi_j v^\beta \right) e_\alpha$$

$$\langle \sigma(L)(\underbrace{x, \xi}_{T^*\mathcal{M}}) v, \underbrace{v}_{\Gamma(E)} \rangle \geq \lambda |\xi|^2 |v|^2$$

II. non-linear quasi-linear

$$P(v) = \left[a^{ij}_{\alpha\beta}(x, v, \nabla v)\, \partial_i \partial_j v^\beta + b^\alpha(x, v, \nabla v) \right] e_\alpha$$

Require $\partial_t v = [DP(w)] v$

parabolic

$$\begin{cases} \partial_t v = P(v) & t \in [0, \varepsilon] \\ v(0) = w \end{cases}$$

$\exists !$

Short time existence

$$\partial_t g = -2\,Ric(g)$$

on $E = Sym^2 T^* M$ $\quad\swarrow$ linearization

$$(D\,Ric(g))(h) = -\frac{1}{2}\Big(\underbrace{\Delta_L}_{}\, h + \mathcal{L}_{(\delta G(h))^{\#}}\, g\Big)$$

symbol +ive def. $\qquad$ not !!

$$\sigma(\Delta_L)(x,\xi)\,h = |\xi|^2 h$$

Step 1. DeTurck's trick

Claim: $\qquad$ flow is parabolic

modified Ricci tensor $\qquad$ elliptic

$$\begin{cases} E_g = 0 \\ \delta G(T) = 0 \end{cases}$$

$$\Updownarrow$$

$$F_g = E_g - \Phi(g,t)$$

$$LHS = D(\partial_t g) = \partial_t h$$

A closer look at $\mathcal{L}_{(\delta G(h))^{\#}}\, g$

$$\mathcal{L}_{(\underbrace{\delta G(e^{s\phi} h)}_{\omega})^{\#}}\, g = \nabla \omega(\underset{1}{\cdot},\underset{2}{\cdot}) + \nabla \omega(\cdot,\cdot)$$

$$= e^{s\phi}\cdot s^2\Big(-\xi \otimes h(\xi^{\#},\cdot) - h(\xi^{\#},\cdot)\otimes \xi + (\xi\otimes\xi)\,\mathrm{tr}\,h\Big)$$

$$+\ e^{s\phi}\cdot \text{l.o.t in } s$$

$$\sigma\Big(\overbrace{h \longmapsto \mathcal{L}_{(\delta G(e^{s\phi}h))^{\#}}\, g}^{\text{operator } L}\Big)\underset{\substack{\cap \\ T^*M}}{(x,\xi)}\ \overset{\nu\in\Gamma(E)}{h}$$

$$= \lim_{s\to\infty} s^{-2} e^{-s\phi} L(e^{s\phi}\nu)(x)$$

$$= -\xi\otimes h(\xi^{\#},\cdot) - h(\xi^{\#},\cdot)\otimes\xi + (\xi\otimes\xi)\,\mathrm{tr}\,h$$

?

$$= |\xi|^2 h - \xi\otimes h(\xi^{\#},\cdot) - h(\xi^{\#},\cdot)\otimes\xi + (\xi\otimes\xi)\,\mathrm{tr}\,h$$

not +ive def.

Need to get rid of $\mathcal{L}_{(\delta G(e^{s\phi}h))^{\#}}\, g$!

Note that

$$\partial_t\big(T^{-1}\delta G(T)\big) = -\delta G(h) + \cdots$$

$$\Rightarrow \partial_t\Big(\mathcal{L}_{(T^{-1}\delta G(T))^{\#}}\, g\Big) = -\mathcal{L}_{(\delta G(T))^{\#}}\, g + \cdots$$

Set $P(t) = -2\,Ric(g(t)) + \mathcal{L}_{(\delta G(T))^\#}\, g(t)$

$= \Delta_L h + h.o.t.$

parabolic

$$\begin{cases} \partial_t g = P(g) \\ g(0) = g_0 \end{cases}$$

Step 2. Modify $\hat{g}(t) = \Psi_t^*(g(t))$

Claim: g satisfies Ricci-deTurck (modified Ricci)

modified $\hat{g}$ satisfies Ricci

pf.

$$X \rightsquigarrow \Psi_t : M \longrightarrow M \text{ diffeo}$$

tension $\tau_{g(t),T}(id)$

$$\partial_t \hat{g} = \Psi_t^*\left(\partial_t g + \mathcal{L}_X g\right)$$

$$= \Psi_t^*\left(-2\,Ric(g) + \mathcal{L}_{(T^{-1}\delta G(T))^\#}\, g + \mathcal{L}_{-(T^{-1}\delta G(T))^\#}\, g\right)$$

$$= \Psi_t^*\left(-2\,Ric(g)\right)$$

$$= -2\,Ric(\hat{g})$$

Uniqueness

- tension field of a map: $\tau(\phi) = \operatorname{tr} \nabla d\phi$
  harmonic if $\tau(\phi) = 0$

- $-(T^{-1} \delta G(T))^{\#} = \tau_{g(t),T}(id) = \operatorname{tr} \nabla d(id)$

where τ tension of

$$(M, g(t)) \xrightarrow{id} (M, T)$$

→ auxiliary metric

Ψ_t is a solution of the harmonic map flow

$$(\star) \qquad \frac{\partial \Psi_t^i}{\partial t} = \tau_{g(t),T}(\Psi_t^i)$$

$g_1(0) = g_2(0) \longrightarrow (\star)$ satisfied for $T_1 = T_2 = T$

small ε $\quad M \times [0,\varepsilon)_t \longrightarrow M$

IC $\quad \Psi^1(0) = \Psi^2(0) = id \quad$ so within small ε

$\Psi^i(t)$ diffeo

$\downarrow$ modified Ricci is diffeo invariant

$\Psi^i_*(g_i(t))$ solves modified Ricci flow with same IC (initial metric $g_1(0) = g_2(0)$)

Follows from uniqueness of parabolic solution

$|Ric| \le M$ on $\mathcal{M} \times [0, S]$

$$e^{-2Mt} g(0) \le g(t) \le e^{2Mt} g(0) \qquad t \in [0, S]$$

prevent the metric from degenerate

<u>Thm</u> (curvature blows up at sing).
$\mathcal{M}$ closed
$g(t)$ Ricci flow on maximal $[0, T)_t$ $\quad T < \infty$.

$$\Rightarrow \quad \sup_{\mathcal{M}} |Rm|(\cdot, t) \longrightarrow \infty \quad \text{as} \quad t \to \infty.$$

pf. suppose not $\rightsquigarrow$ can extend g to $[0, T+\varepsilon)$

Step 1.
$|Rm| \le M$ at $t_0 = T - \varepsilon$ then by Thm 3.2.11

$$|Rm| \le \frac{M}{1 - CM(t - t_0)} \qquad t \in [t_0, T]$$

$\rightsquigarrow$ allows to extend g up to $g(T)$

Step 2. $\quad$ initial metric
$\quad$ + short time existence $\quad g(t),\ t \in [0, \underline{T+\varepsilon})$

contradicting T max.

smoothness of $g(t)$ follows from maximal principle